THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 4

§30) 1) b) $e^{\frac{2+\pi i}{4}} = e^{\frac{1}{2}} e^{i\frac{\pi}{4}} = \sqrt{e}\left(\frac{1+i}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}}(1+i)$.

c) $e^{z+\pi i} = e^z e^{\pi i} = -e^z$.

§30) 2) Since addition, substaction, multiplication and composition of enitre functions is entire, the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

§30) 5) Note that

$$|e^{2z+i}| = |e^{2x+(2y+1)i}| = e^{2x} \text{ and } |e^{iz^2}| = |e^{i(x^2-y^2+2xyi)}| = |e^{-2xy+i(x^2-y^2)}| = e^{-2xy}.$$

Therefore, by triangle inequality,

$$|e^{2z+i} + e^{iz^2}| \leq |e^{2z+i}| + |e^{iz^2}| = e^{2x} + e^{-2xy}.$$

§38) 2) a) Since $e^{iz} = \cos z + i \sin z$, we have

$$\begin{aligned} e^{iz_1}e^{iz_2} &= (\cos z_1 + i\sin z_1)(\cos z_2 + i \sin z_2) \\ &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \end{aligned}$$

Replace z_1 and z_2 by $-z_1$ and $-z_2$ respectively, we have

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

b) Since $\sin(z_1 + z_2) = \frac{1}{2i}(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2})$, we have

$$\begin{aligned} \sin(z_1+z_2) &= \frac{1}{2i}(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}) \\ &= \frac{1}{2i}\left[(\cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2))\right. \\ &\quad \left. - (\cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2))\right] \\ &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2. \end{aligned}$$

§38) 8) Since $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$, we have

$$\begin{aligned} |\sin z| &= \sqrt{|\sin z|^2} = \sqrt{\sin^2 x + \sinh^2 y} \geq \sqrt{\sin^2 x} = |\sin x| \text{ and} \\ |\cos z| &= \sqrt{|\cos z|^2} = \sqrt{\cos^2 x + \sinh^2 y} \geq \sqrt{\cos^2 x} = |\cos x|. \end{aligned}$$

§38) 15)

$$\begin{aligned} && \sin z &= \cosh 4 \\ \Longrightarrow && \sin x \cosh y + i \cos x \sinh y &= \cosh 4 \\ \Longrightarrow && \sin x \cosh y = \cosh 4 &\text{ and } \cos x \sinh y = 0 \end{aligned}$$

Note that if $\sinh y = 0$, we have $y = 0$ and $\cosh 4 = \sin x \cosh y = \sin x < 1$, which is impossible. Hence we must have $\cos x = 0$ and hence $x = n\pi + \frac{\pi}{2}$ for some $n \in \mathbb{Z}$. Since $\sin x \cosh y = \cosh 4 > 0$, we have $x = 2n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$.

As a result, we have $\cosh y = \cosh 4$. Since the function $\cosh y$ is strictly increasing on $(0, \infty)$ and it is an even function, we have $y = \pm 4$. Altogether, we have $z = \left(\frac{\pi}{2} + 2n\pi\right) \pm 4i$, $n \in \mathbb{Z}$.

§39) 1) Note that

$$\begin{aligned}
\frac{d}{dz}\sinh z &= \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{1}{2}\left(\frac{d}{dz}e^z - \frac{d}{dz}e^{-z}\right) = \frac{e^z + e^{-z}}{2} = \cosh z \\
\frac{d}{dz}\cosh z &= \frac{d}{dz}\frac{e^z + e^{-z}}{2} = \frac{1}{2}\left(\frac{d}{dz}e^z + \frac{d}{dz}e^{-z}\right) = \frac{e^z - e^{-z}}{2} = \sinh z.
\end{aligned}$$

§39) 2) To show that $\sinh 2z = 2\sinh z \cosh z$,

a) note that

$$\begin{aligned}
\text{L.H.S.} &= \sinh 2z = \frac{e^{2z} - e^{-2z}}{2} \\
\text{R.H.S.} &= 2\sinh z \cosh z = 2\left(\frac{e^z - e^{-z}}{2}\right)\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^{2z} - e^{-2z}}{2} = \text{L.H.S.}
\end{aligned}$$

b) Since $\sin 2z = 2\sin z \cos z$, we have

$$\begin{aligned}
& & \sin 2(-iz) &= 2\sin(-iz)\cos(-iz) \\
&\Longrightarrow & -i\sinh 2z &= 2(-i\sinh z)(\cosh(z)) \\
&\Longrightarrow & \sinh 2z &= 2\sinh z \cosh z.
\end{aligned}$$

§39) 8) Note that

$$\begin{aligned}
\sinh z = 0 &\iff |\sinh z|^2 = 0 \\
&\iff \sinh^2 x + \sin^2 y = 0 \\
&\iff \sinh x = 0 = \sin y \\
&\iff x = 0 \text{ and } y = n\pi, n \in \mathbb{Z} \\
&\iff z = n\pi i, n \in \mathbb{Z}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cosh z = 0 &\iff |\cosh z|^2 = 0 \\
&\iff \sinh^2 x + \cos^2 y = 0 \\
&\iff \sinh x = 0 = \cos y \\
&\iff x = 0 \text{ and } y = \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \\
&\iff z = \left(\frac{\pi}{2} + n\pi\right) i, n \in \mathbb{Z}.
\end{aligned}$$